

A remark on the motivic Galois group and the quantum coadjoint action

H. Grosse^a, K.-G. Schlesinger^b

Institute for Theoretical Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria

Received: 13 October 2005 /

Published online: 13 March 2006 – © Springer-Verlag / Società Italiana di Fisica 2006

Abstract. It has been suggested that the Grothendieck–Teichmüller group GT should act on the Duflo isomorphism of $su(2)$, but the corresponding realization of GT turned out to be trivial. We show that a solvable quotient of the motivic Galois group – which is supposed to agree with GT – is closely related to the quantum coadjoint action on $U_q(sl_2)$ for q a root of unity, i.e. in the quantum group case one has a nontrivial realization of a quotient of the motivic Galois group. From a discussion of the algebraic properties of this realization we conclude that in more general cases than $U_q(sl_2)$ it should be related to a quantum version of the motivic Galois group. Finally, we discuss the relation of our construction to quantum field and string theory and explain what we believe to be the ‘physical reason’ behind this relation between the motivic Galois group and the quantum coadjoint action. This might be a starting point for the generalization of our construction to more involved examples.

1 Introduction

In the seminal paper [23], a symmetry on the space $\mathcal{D}(M)$ of deformation quantizations of a finite-dimensional manifold M , in the form of an action of a quotient of the motivic Galois group (this quotient is supposed to be equivalent to the Grothendieck–Teichmüller group GT as introduced in [9]), is discussed. Conjecturally, this is related to an action of GT on the extended moduli space (see [21, 32] for this notion) of conformal field theories, as it appears in string theory. As a simple example for the appearance of this symmetry, an action of GT on the Duflo isomorphism of finite-dimensional Lie algebras is suggested. But, it was observed by Duflo (see [23]) that the corresponding realization of GT is trivial.

In this paper, we restrict the consideration to the case of the Lie algebra $su(2)$. We pose the question of whether a nontrivial realization of the motivic Galois group can be observed in the q -deformed case. Since our argument does not involve any $*$ -structure, we can work with the quantum algebra $U_q(sl_2)$. We show that in the case where q is a root of unity, the quantum coadjoint action on $U_q(sl_2)$ (see [7, 8]) is, indeed, closely related to a quotient of the motivic Galois group, which is stated in [23] to act on the Hochschild cohomology of algebraic varieties and is conjectured, there, to be equivalent to GT. It is precisely the much larger and highly nontrivial center of $U_q(sl_2)$, appearing in the root of unity case, which makes a nontrivial

realization of the motivic Galois group possible. This explains why one observes in the nondeformed case – but also in the case of generic values of q – only a trivial realization.

After collecting, in Sect. 2, some basic background material on the motivic Galois group and the quantum coadjoint action, we present, in Sect. 3, our construction. In Sect. 4, we discuss the relation of our result to quantum field and string theory. The algebraic structure appearing in our construction suggests that in the case of more involved examples than $U_q(sl_2)$ a quantum version of the motivic Galois group will appear. We suggest a physical ‘explanation’, rooted in properties of quantum field theory, of why our construction works, which might serve as a starting point for the generalization to other examples. Section 5 contains some concluding remarks.

We should utter a warning addressed to any potential reader of this paper who is on a technical level acquainted with modern algebraic number theory: the necessary specification of the precise type of motivic Galois group to use (pro-nilpotent, pro-unipotent, etc.) is completely ignored in this paper. We work with the (pro-unipotent) approach of [23] and feel free to adopt the terminology ‘the motivic Galois group’ used there.

2 Background material on the motivic Galois group and the quantum coadjoint action

The Grothendieck–Teichmüller group GT is introduced in [9] as a kind of gauge freedom on the Drinfeld associator α and the R -matrix of any quasitriangular quasi-Hopf

^a e-mail: grosse@doppler.thp.univie.ac.at

^b e-mail: kgschles@esi.ac.at

algebra (see [9] for the technical details). It was already observed there that the Lie algebra of GT is closely related to the so-called Ihara algebra (see [18, 19]). The Ihara algebra has the following structure: consider formal expressions $\varphi(\cdot)$ where the (\cdot) indicates that these expressions can be evaluated at any finite-dimensional metrizable – i.e. equipped with an invariant inner product – Lie algebra g . After evaluation, $\varphi(g)$ becomes an element of the Poisson algebra defined by the Kirillov bracket $\{, \}_g$ of g . So, the $\varphi(\cdot)$ are, roughly speaking, all universal expressions which one can define for any finite-dimensional metrizable Lie algebra using the Kirillov bracket. The Ihara algebra is then defined as the Lie algebra with bracket $[,]$ on the $\varphi(\cdot)$ with

$$[\varphi_1, \varphi_2](g) = \{\varphi_1(g), \varphi_2(g)\}_g. \tag{1}$$

It is conjectured in [23] that GT can be identified with a solvable quotient of the motivic Galois group, the Lie algebra of this quotient having generators $L_0, P_3, P_5, P_7, \dots$ and the bracket being given by

$$[P_{2k+1}, P_{2l+1}] = 0 \tag{2}$$

and

$$[L_0, P_{2k+1}] = (2k + 1) P_{2k+1} \tag{3}$$

for all $k, l \geq 1$. A proof is announced there that this quotient acts on the Hochschild cohomology of any complex algebraic variety.

A simple explicit example for an action of GT is suggested in [23] by considering the Duflo isomorphism of a finite-dimensional Lie algebra g over \mathbb{R} : the Poincaré–Birkhoff–Witt isomorphism gives a linear isomorphism between the universal envelope $U(g)$ of g and the algebra $Sym(g)$ of polynomials on g^* . This is, obviously, not an algebra isomorphism since $U(g)$ is noncommutative. But, as shown in [10], the restriction of the Poincaré–Birkhoff–Witt map to a linear isomorphism between the center of $U(g)$ and the algebra $Sym(g)^g$ of invariant polynomials on g^* becomes an algebra isomorphism after combining it with an automorphism of $Sym(g)^g$. This automorphism is generated (see e.g. [22] for the technical details) by the formal power series

$$F(x) = \sqrt{\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{x}}. \tag{4}$$

The claim is that there are other possible choices for F than the classical choice (4) and that this ‘gauge freedom’ of the Duflo isomorphism is described by an action of GT. But, this action of GT turns out to be trivial: it was observed by Duflo (see [23]) that, though the different choices of F generate different morphisms of $Sym(g)$, upon restriction to $Sym(g)^g$ all choices are equivalent.

We will show in this paper for the case $g = su(2)$ that upon passing to the quantum algebra $U_q(sl_2)$ for q a root of unity, one does obtain a nontrivial action of the quotient of the motivic Galois group determined by (2) and (3). Since we do not need any $*$ -structure in our construction,

we can work with the algebra $U_q(sl_2)$ instead of a quantum version of $su(2)$. We will close this section with a brief introduction to $U_q(sl_2)$ and the quantum coadjoint action, and a short explanation of why the question of an action of the motivic Galois group related to the center of $U(g)$ or $U_q(g)$ is of special interest in string theory.

The quantum algebra $U_q(sl_2)$, $q \in \mathbb{C}$ is defined as the complex, associative, unital algebra with generators e, f, k, k^{-1} and relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1 \\ ke &= q^2ek \\ kf &= q^{-2}fk \\ [e, f] &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

In addition, $U_q(sl_2)$ carries a uniquely given Hopf algebra structure with coproduct

$$\begin{aligned} \Delta(e) &= e \otimes 1 + k \otimes e \\ \Delta(f) &= f \otimes k^{-1} + 1 \otimes f \\ \Delta(k) &= k \otimes k. \end{aligned}$$

counit

$$\begin{aligned} \varepsilon(e) &= \varepsilon(f) = 0 \\ \varepsilon(k) &= 1. \end{aligned}$$

and antipode

$$\begin{aligned} \Gamma(e) &= -k^{-1}e \\ \Gamma(f) &= -fk \\ \Gamma(k) &= k^{-1}. \end{aligned}$$

For q not a root of unity, i.e. $q^n \neq 1$ for all $n \in \mathbb{N}$, the center of $U_q(sl_2)$ is generated by the quantum Casimir C_q given by

$$C_q = qk + q^{-1}k^{-1} + (q - q^{-1})^2 fe. \tag{5}$$

Now, let q be a primitive l th root of unity with $l \geq 3$, i.e. $q^l = 1$ and $q^n \neq 1$ for $n < l$. In this case, the center of $U_q(sl_2)$ is generated by C_q and Z_0 , where Z_0 is the commutative subalgebra of $U_q(sl_2)$ generated by the elements x, y, z, z^{-1} with

$$\begin{aligned} x &= ((q - q^{-1})e)^{\bar{l}} \\ y &= ((q - q^{-1})f)^{\bar{l}} \\ z &= k^{\bar{l}} \\ z^{-1} &= k^{-\bar{l}} \end{aligned}$$

and

$$\bar{l} = l$$

for l odd, and

$$\bar{l} = \frac{l}{2}$$

for l even. For a more detailed introduction to $U_q(sl_2)$ we refer the reader to [5] or [26].

The quantum coadjoint action (see [7, 8]) on $U_q(sl_2)$ is introduced as follows: define derivations \underline{e} , \underline{f} , \underline{k} , \underline{k}^{-1} on $U_q(sl_2)$ by

$$\begin{aligned} \underline{e}(a) &= \left[\frac{e^{\bar{l}}}{[\bar{l}]_q!}, a \right] \\ \underline{f}(a) &= \left[\frac{f^{\bar{l}}}{[\bar{l}]_q!}, a \right] \\ \underline{k}(a) &= \left[\frac{k^{\bar{l}}}{[\bar{l}]_q!}, a \right] \\ \underline{k}^{-1}(a) &= \left[\frac{(k^{-1})^{\bar{l}}}{[\bar{l}]_q!}, a \right] \end{aligned}$$

for $a \in U_q(sl_2)$. Remember that the q -factorial $[n]_q!$ is given by

$$[n]_q! = [n]_q [n-1]_q \dots [1]_q .$$

These derivations stay well defined in the limit $q^l \rightarrow 1$ and can, alternatively, be defined by

$$\begin{aligned} \underline{e}(e) &= 0 \\ \underline{e}(f) &= \frac{kq - k^{-1}q^{-1}}{q - q^{-1}} \frac{e^{l-1}}{[l-1]!} \\ \underline{e}(k) &= -l^{-1}xk \\ \underline{e}(k^{-1}) &= l^{-1}xk^{-1} \end{aligned}$$

and

$$\begin{aligned} \underline{f}(e) &= -\frac{f^{l-1}}{[l-1]!} \frac{kq - k^{-1}q^{-1}}{q - q^{-1}} \\ \underline{f}(f) &= 0 \\ \underline{f}(k) &= l^{-1}yk \\ \underline{f}(k^{-1}) &= -l^{-1}yk^{-1} . \end{aligned}$$

We only give this alternative definition for \underline{e} , \underline{f} here, since these will serve as the generators of the quantum coadjoint action (see below). The action of \underline{e} , \underline{f} , \underline{k} , \underline{k}^{-1} on the subalgebra Z_0 is given by

$$\begin{aligned} \underline{e}(x) &= 0 \\ \underline{e}(y) &= z - z^{-1} \\ \underline{e}(z) &= -xz \\ \underline{e}(z^{-1}) &= -xz^{-1} \end{aligned}$$

respectively

$$\begin{aligned} \underline{f}(x) &= -(z - z^{-1}) \\ \underline{f}(y) &= 0 \\ \underline{f}(z) &= yz \\ \underline{f}(z^{-1}) &= -yz^{-1} \end{aligned}$$

respectively

$$\begin{aligned} \underline{k}(x) &= xz \\ \underline{k}(y) &= -yz \\ \underline{k}(z) &= \underline{k}(z^{-1}) = 0 \end{aligned}$$

and

$$\begin{aligned} \underline{k}^{-1}(x) &= xz^{-1} \\ \underline{k}^{-1}(y) &= -yz^{-1} \\ \underline{k}^{-1}(z) &= \underline{k}^{-1}(z^{-1}) = 0 . \end{aligned}$$

Exponentiating \underline{e} and \underline{f} yields automorphisms of $U_q(sl_2)$. We denote by \mathcal{G} the (infinite-dimensional) group of automorphisms of $U_q(sl_2)$ generated by the exponentials of \underline{e} and \underline{f} and by $\mathcal{L}(\mathcal{G})$ the Lie algebra of \mathcal{G} .

Now, let $\widehat{\mathcal{G}}$ be the following group: as elements of $\widehat{\mathcal{G}}$ we take formal expressions $\varphi(\cdot)$, which can be evaluated at any Hopf algebra H in the class of Hopf algebras that carry a quantum coadjoint action. Denote by \mathcal{G}_H the corresponding group of automorphisms, i.e.

$$\mathcal{G}_{U_q(sl_2)} = \mathcal{G} .$$

We assume that after evaluation

$$\varphi(H) \in \mathcal{G}_H .$$

Define the group law of $\widehat{\mathcal{G}}$ by

$$(\varphi_1 \cdot \varphi_2)(H) = \varphi_1(H) \cdot \varphi_2(H)$$

where on the right-hand side the multiplication is taken in \mathcal{G}_H . It is easily checked by calculation that this together with the definition of a unit 1 and an inverse $(\cdot)^{-1}$ of $\widehat{\mathcal{G}}$ by

$$1(H) = 1_{\mathcal{G}_H}$$

and

$$\varphi^{-1}(H) = (\varphi(H))^{-1}$$

gives $\widehat{\mathcal{G}}$ the structure of a group.

Obviously, $\widehat{\mathcal{G}}$ generalizes the Ihara algebra in precisely the same way in which the quantum coadjoint action generalizes the classical coadjoint action and the Kirillov bracket. In this sense, the quantum coadjoint action seems to be a concrete realization of a universal structure related to a quantum counterpart of the motivic Galois group. We will see in Sect. 3 that one can establish this claim for $U_q(sl_2)$ in a precise way by studying the Lie algebra $\mathcal{L}(\mathcal{G})$ in more detail.

Let us close this section by briefly mentioning the role played by $U_q(sl_2)$ in string theory: for a stack of k flat NS5 branes the background is completely determined by vanishing Ramond–Ramond (R–R) fields and (see [3, 4, 12, 30])

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi} (dr^2 + r^2 ds_3^2) \\ e^{-2\phi} &= e^{-2\phi_0} \left(1 + \frac{k}{r^2} \right) \\ H &= dB = -k d\Omega_3 \end{aligned}$$

where $\mu, \nu = 0, 1, \dots, 5$ are directions tangent to the NS5 brane. Here, ds_3 and $d\Omega_3$ denote the line element and volume form, respectively, on the S^3 . So, in the transversal geometry of the flat NS5 there is always contained an S^3 . Strings on the transversal S^3 can always be described by a super-extension of the $SU(2)$ -WZW model at level k . One can further show (see the cited literature) that the fermionic degrees of freedom can be decoupled and one is left with the usual $SU(2)$ -WZW model and a renormalization of the level k by

$$k \mapsto k + 2.$$

It has further been shown in [1] that the world-volume geometry of D -branes in the $SU(2)$ -WZW model at level k is described by the q -deformed fuzzy sphere of [15], i.e. by the representation theory of $U_q(sl_2)$ with

$$q = e^{\frac{2\pi i}{k+2}}.$$

In the limit $k \rightarrow \infty$ one retains the usual fuzzy sphere of [28], which is completely determined by the representation theory of $su(2)$.

In [23] and [25] a far-reaching program was initiated to establish an action of (a quotient of) the motivic Galois group on the extended moduli space of (topological) string theory. It is tempting to ask for a similar approach in the much simpler case of the $SU(2)$ -WZW model. The suggestion made in [23] to study an action of the motivic Galois group on the center of $U(su(2))$ (more precisely, on the Duflo isomorphism of $su(2)$, see above) is very much related to trying to find such an action for the $SU(2)$ -WZW model at level $k \rightarrow \infty$. As we have mentioned already, the action one finds in this way is trivial. Now, $k \rightarrow \infty$ is an unphysical limit (infinite stack of NS5 branes). The approach we follow, here, to study the center of the quantum algebra $U_q(sl_2)$ at roots of unity, instead, can from the string theoretic side be seen as returning to the physically more realistic case of finite level k .

3 The construction

As is well known, the Lie algebra $\mathcal{L}(\mathcal{G})$ is infinite dimensional. Concretely, this means that the commutator $[\underline{e}, \underline{f}]$ does not close but is the starting point for the definition of infinitely many new basis elements of the Lie algebra. Here, and in the sequel, we mean by a commutator like $[\underline{e}, \underline{f}]$ the commutator in the sense of derivations, i.e.

$$[\underline{e}, \underline{f}] = \underline{e} \circ \underline{f} - \underline{f} \circ \underline{e}$$

where \circ denotes the successive application of the derivations. We introduce the following notation:

$$\begin{aligned} L &= \underline{k} + \underline{k}^{-1} \\ e_0 &= \underline{e} \\ f_0 &= \underline{f}. \end{aligned}$$

One checks that

$$[\underline{e}, \underline{f}] = L.$$

We further define for all $N \in \mathbb{N}$

$$\begin{aligned} e_{N+1} &= [L, e_N] \\ f_{N+1} &= [L, f_N]. \end{aligned}$$

Lemma 1. *On the subalgebra Z_0 we have*

$$\begin{aligned} e_N(x) &= Nx^2 (z + z^{-1})^{N-1} (z - z^{-1}) \\ e_N(y) &= (z + z^{-1})^N (z - z^{-1}) \\ &\quad - Nxy (z + z^{-1})^{N-1} (z - z^{-1}) \\ e_N(z + z^{-1}) &= (-x) (z + z^{-1})^N (z - z^{-1}) \end{aligned}$$

and

$$\begin{aligned} f_N(x) &= (-1)^{N-1} (z + z^{-1})^N (z - z^{-1}) \\ &\quad + (-1)^N Nxy (z + z^{-1})^{N-1} (z - z^{-1}) \\ f_N(y) &= (-1)^{N-1} Ny^2 (z + z^{-1})^{N-1} (z - z^{-1}) \\ f_N(z + z^{-1}) &= (-1)^N y (z + z^{-1})^N (z - z^{-1}). \end{aligned}$$

Proof. By calculation.

For convenience, we have chosen the coordinate $z + z^{-1}$ instead of z (observe that z and z^{-1} are not independent, i.e. we do not have to take the complementary coordinate $z - z^{-1}$, in addition).

Let \mathcal{A} be the algebra of polynomials in z and z^{-1} . Defining

$$L_0 = z \frac{d}{dz}$$

and

$$\overline{P}_{2k+1} = z^{2k+1}$$

one checks that for arbitrary $\psi \in \mathcal{A}$ we have

$$\begin{aligned} &[L_0, \overline{P}_{2k+1}] \psi \\ &= L_0 \overline{P}_{2k+1} \psi - \overline{P}_{2k+1} L_0 \psi \\ &= \left(z \frac{d}{dz} \overline{P}_{2k+1} \right) \psi + z \overline{P}_{2k+1} \left(\frac{d}{dz} \psi \right) - z \overline{P}_{2k+1} \left(\frac{d}{dz} \psi \right) \\ &= (2k+1) \overline{P}_{2k+1} \psi \end{aligned}$$

i.e.

$$[L_0, \overline{P}_{2k+1}] = (2k+1) \overline{P}_{2k+1}$$

and L_0 and the \overline{P}_{2k+1} give a representation of the algebra defined by (2) and (3). Denote this representation by ρ .

Let I and A be the following two linear operators on \mathcal{A} :

$$I(\psi) = z\psi$$

and

$$A(z) = z + z^{-1}$$

for $\psi \in \mathcal{A}$ and extension of A to \mathcal{A} by requiring conservation of products and inversion.

Obviously, I commutes with the \overline{P}_{2k+1} . We can use I to enlarge ϱ by considering the additional commutators

$$[L_0, I\overline{P}_{2k+1}] = (2k+2)I\overline{P}_{2k+1}$$

i.e. the resulting algebra has the form

$$[L_0, \overline{P}_n] = n\overline{P}_n$$

where now $n \in \mathbb{N}$. Finally, we use A to apply a coordinate transformation to the operator \overline{P}_n :

$$\overline{P}_n \mapsto \overline{P}_n \circ A.$$

We obtain the resulting commutators

$$[L_0, P_n] = nP_{n-1}(z - z^{-1}) \tag{6}$$

where

$$P_n = \overline{P}_n \circ A$$

i.e.

$$P_n(z) = (z + z^{-1})^n.$$

We call the Lie algebra defined by (6) $\widehat{\varrho}$. As follows from the construction, $\widehat{\varrho}$ is directly induced by a coordinate transformation on one of the operators and a trivial enlargement by the operator I from the representation ϱ .

We can now rewrite the result of the previous lemma as

$$\begin{aligned} e_N(z + z^{-1}) &= (-x)(z + z^{-1})[L_0, P_N](z) \\ e_N(x) &= Nx^2[L_0, P_N](z) \\ e_N(y) &= (z + z^{-1} - Nxy)[L_0, P_N](z) \end{aligned}$$

and

$$\begin{aligned} f_N(z + z^{-1}) &= (-1)^{N-1}(-y)(z + z^{-1})[L_0, P_N](z) \\ f_N(x) &= (-1)^{N-1}(z + z^{-1} - Nxy)[L_0, P_N](z) \\ f_N(y) &= (-1)^{N-1}Ny^2[L_0, P_N](z). \end{aligned}$$

As we have remarked already, from the four variables x, y, z, z^{-1} , z and z^{-1} are not independent. Actually, only two free variables remain after taking a quotient to implement the relation

$$z + z^{-1} + xy = 0. \tag{7}$$

This should be done because of the following lemma:

Lemma 2. *For all $N \in \mathbb{N}$ we have*

$$e_N(z + z^{-1} + xy) = f_N(z + z^{-1} + xy) = 0.$$

Proof. We have

$$\begin{aligned} &e_N(z + z^{-1} + xy) \\ &= (-x)(z + z^{-1})^N(z - z^{-1}) + e_N(x)y + xe_N(y) \\ &= (-x)(z + z^{-1})^N(z - z^{-1}) \\ &\quad + Nx^2y(z + z^{-1})^{N-1}(z - z^{-1}) + x(z + z^{-1})^N(z - z^{-1}) \\ &\quad - Nx^2y(z + z^{-1})^{N-1}(z - z^{-1}) \\ &= 0. \end{aligned}$$

A similar calculation proves the case of the f_N .

On a more abstract level, the result of the previous lemma can be seen as a consequence of the fact that the quantum Casimir C_q is invariant under the quantum coadjoint action, i.e. the polynomial $z + z^{-1} + xy$ is annihilated by $\mathcal{L}(\mathcal{G})$ (see the cited literature; see also Sect. 2 of [24] for a brief overview of some properties of the quantum coadjoint action).

Using (7), we finally have

$$\begin{aligned} e_N(z + z^{-1}) &= (-x)(z + z^{-1})[L_0, P_N](z) \\ e_N(x) &= Nx^2[L_0, P_N](z) \\ e_N(y) &= -(N+1)xy[L_0, P_N](z) \end{aligned} \tag{8}$$

and

$$\begin{aligned} f_N(z + z^{-1}) &= (-1)^{N-1}(-y)(z + z^{-1})[L_0, P_N](z) \\ f_N(x) &= (-1)^{N-1}(-(N+1))xy[L_0, P_N](z) \\ f_N(y) &= (-1)^{N-1}Ny^2[L_0, P_N](z). \end{aligned} \tag{9}$$

In conclusion, the operators e_N, f_N give a three-dimensional – using variables x, y, z – realization of the algebra $\widehat{\varrho}$. The nontrivial action of these operators on the variable z is always given by a commutator in $\widehat{\varrho}$. The additional polynomial in x, y, z, z^{-1} in front of the commutator is completely determined – up to a numerical factor – by the trivial rule that e_N multiplies the argument by x and f_N multiplies the argument by y . For example, $e_N(z + z^{-1})$ receives the factor $x(z + z^{-1})$ while $f_N(x)$ receives the factor xy . One may then prove that the numerical coefficients are completely determined by the requirement (7).

As an immediate consequence of (8) and (9), the commutators $[e_N, e_M], [e_N, f_M]$, etc. are determined by the higher commutators in $\widehat{\varrho}$ together with an extension of the multiplication rule for the coefficients. For example, $[e_N, e_M](y)$ means that the coefficient polynomial is x^2y since we have two factors e_N, e_M and, hence, a multiplication by x^2 . In consequence, we have the following result:

Lemma 3. *The complete algebra $\mathcal{L}(\mathcal{G})$ is induced, in the way described above, by a representation of the solvable quotient of the motivic Galois group given by (2) and (3).*

The nontriviality of the quantum coadjoint action immediately shows that the quotient of the motivic Galois group is realized nontrivially on $U_q(\mathfrak{sl}_2)$.

Remark 1. The coefficient factors obtained by the rule that e_N multiplies by x and f_N by y strongly remind

one of how one would introduce an affine version of the algebra given by (2) and (3). In the case of the classical finite-dimensional Lie algebras, the theorem of Kazhdan–Lusztig [27] relates the affine version of the Lie algebra to a quantum deformation of the universal envelope of the Lie algebra. A corresponding theorem for the infinite-dimensional case is not proved but we will see below that, indeed, we should see $\mathcal{L}(\mathcal{G})$ as a realization of a quantum counterpart of the motivic Galois group. This is also in accordance with the construction of $\widehat{\mathcal{G}}$ above as a quantum analogue of the Ihara algebra. The fact that in the case of $U_q(\mathfrak{sl}_2)$ we mainly – up to a simple multiplication table – see the classical motivic Galois group should be related to the semi-rigidity of $U_q(\mathfrak{sl}_2)$ (i.e. one can always transform away deformations of the coproduct), which makes the Hopf algebra cohomology of $U_q(\mathfrak{sl}_2)$ very close to Hochschild cohomology.

In Sect. 4, we will give a physics-motivated ‘explanation’ for the above results, which also points towards their possible generalization.

4 The physics behind

In [23, 25] the action of GT on Hochschild cohomology is considered. If one passes from Hochschild to Hopf algebra cohomology, one expects a doubling of this GT action with precisely the compatibility relations installed which lead to the definition of the self-dual, noncommutative, and non-cocommutative Hopf algebra \mathcal{H}_{GT} (see [31]).

As we have mentioned in Sect. 2, $U_q(\mathfrak{sl}_2)$ appears in string theory as describing the world-volume geometry of D -branes in the $SU(2)$ -WZW model. We now make

Assumption 1: The generators of a suitable moduli space of the boundary $SU(2)$ -WZW model should be given by the Hopf algebra cohomology of $U_q(\mathfrak{sl}_2)$.

With this assumption, we can conclude that the deformation theory of the boundary $SU(2)$ -WZW model should have an action of \mathcal{H}_{GT} as a symmetry.

It is one of the general beliefs in the theory of the quantum coadjoint action (see e.g. [5]) that – roughly speaking – the following holds true:

Assumption 2: The representation theory of a Hopf algebra with quantum coadjoint action is determined by the quantum coadjoint action.

In physics terminology, this means especially that in the $SU(2)$ -WZW model the operator product expansion should be determined by knowing \mathcal{G} .

If we now make Assumption 3, below, we can conclude that the quantum coadjoint action should give a realization of \mathcal{H}_{GT} . In the semi-rigid case of $U_q(\mathfrak{sl}_2)$ – where we do not really have the doubling of GT to \mathcal{H}_{GT} – this means that the quantum coadjoint action should give a realization of the quotient of the motivic Galois group, which is supposed to be equivalent to GT. This is precisely what we have shown in Sect. 3.

Assumption 3: The $SU(2)$ -WZW model should be *formal* in the sense that its deformation theory should agree with its algebra of observables.

This formality assumption (the cohomology of the algebraic structure should agree with the algebraic structure itself) is the essential input to pass from \mathcal{H}_{GT} to the quantum coadjoint action. We believe that this is a key element to generalize our approach: quantum field theories which have such a formality property should give a realization of \mathcal{H}_{GT} in terms of a generalization of the quantum coadjoint action on $U_q(\mathfrak{sl}_2)$.

There are indications that in many cases this formality assumption holds true in quantum field theory: for two dimensional conformal field theories (2d CFTs) the deformations described by the WDVV equations (see [11, 33]) are, indeed, in one to one correspondence to the observables of the theory. If one turns on background fields or introduces NS-branes in string theory, the deformation theory becomes much more complicated (see the fundamental work [17]). We have suggested in [16] a universal envelope for the BRST-complex, the cohomology of which might describe the full deformation theory with background fields and NS-branes. On the level of this universal envelope we, again, expect a formality property to hold as a consequence of ultrarigidity (see [31] for this notion).

So, Assumption 3 might well be a general property of quantum field theory, but in more complicated models it might only hold true if one includes all the allowed background fields into the model. Turned the other way around, Assumption 3 could serve as a guideline to search for necessary background fields to include in order to satisfy this principle. If such a view holds true, our construction might be an example of a general link between \mathcal{H}_{GT} as a quantum counterpart of the motivic Galois group and its representation-theoretic realization in quantum field theory.

Let us conclude this section by making three remarks on how we think one could concretely start to generalize our approach to more complicated backgrounds in string theory:

- One should try to exploit the link between D -brane world-volume geometry in the $SU(2)$ -WZW model and NS5-brane backgrounds to obtain knowledge about quantum motivic structures in the case of more general NS5-brane backgrounds and for the case of little string theory (see also the remarks in [16] on this topic).
- In [2] a link between cohomology of tilting modules over quantum groups at roots of unity and derived categories of coherent sheaves has been shown. This might serve as a starting point to study quantum motivic symmetries on these derived categories and could therefore be highly relevant for the homological mirror symmetry program [21].
- Last but not least, the quantum coadjoint action on $U_q(\mathfrak{sl}_2)$ has been shown to relate to a hidden symmetry of the six-vertex model (see [6, 13, 14, 24]). Can one generalize this approach to the melting crystal models appearing in topological string theory (see [20, 29]).

This might be a very direct way to establish a large hidden symmetry of a quantum motivic type in topological string theory.

5 Conclusion

We have shown that the Lie algebra $\mathcal{L}(\mathcal{G})$ of the quantum coadjoint action on $U_q(sl_2)$ is induced by a representation of the solvable quotient of the motivic Galois group given by (2) and (3). We have also given an argument as to why we think that our construction might relate to a general formality property of quantum field theories and pointed out different directions in which to achieve such a generalization.

We plan to study the physical implications of this link between the quantum coadjoint action and the motivic Galois group in more detail in the near future.

Acknowledgements. KGS would like to thank the Technical University of Vienna for support under the FWF project P15553 ‘Noncommutative structures in open string theory’.

References

1. A.Y. Alekseev, A. Recknagel, V. Schomerus, hep-th/9908040v2
2. R. Bezrukavnikov, math.RT/0403003
3. M. Bianchi, Y.S. Stanev, hep-th/9711069
4. C.G. Callan, J.A. Harvey, A. Strominger, Nucl. Phys. B **359**, 611 (1991)
5. V. Chari, A. Pressley, Quantum Groups (Cambridge University Press, Cambridge, 1994)
6. T. Deguchi, K. Fabricius, B.M. McCoy, J. Statist. Phys. **102**, 701 (2001) [cond-mat/9912141]
7. C. De Concini, V. Kac, Prog. Math. **92**, 471 (1990)
8. C. De Concini, V. Kac, C. Procesi, J. Am. Math. Soc. **5**, 151 (1992)
9. V.G. Drinfeld, Leningr. Math. J. **2**, 829 (1991)
10. M. Duflo, C. R. Acad. Sci. A **269**, 437 (1969)
11. R. Dijkgraaf, E. Verlinde, H. Verlinde, Nucl. Phys. B **352**, 59 (1991)
12. S. Förste, D. Ghoshal, S. Panda, Phys. Lett. B **411**, 46 (1997) [hep-th/9706057]
13. K. Fabricius, B.M. McCoy, J. Statist. Phys. **103**, 647 (2001) [cond-mat/0009279]
14. K. Fabricius, B.M. McCoy, J. Statist. Phys. **104**, 573 (2001) [cond-mat/0012501]
15. H. Grosse, J. Madore, H. Steinacker, hep-th/0005273v2
16. H. Grosse, K.-G. Schlesinger, hep-th/0412161
17. C. Hofman, W.K. Ma, hep-th/0102201v3
18. Y. Ihara, Some Problems on Three-point Ramifications and Associated Large Galois Representations (Adv. Stud. Pure Math. bf 12) (North-Holland, Amsterdam and New York, 1987), pp.173–188
19. Y. Ihara, The Galois Representations Arising From $\mathbb{P}^1 - \{0, 1, \infty\}$ and Tate Twists of Even Degree, Galois Groups Over \mathbb{Q} (Math. Sci. Res. Inst. Publ. **16**) (Springer, Berlin New York, 1989), pp.299–313
20. A. Iqbal, N. Nekrasov, A. Okounkov, C. Vafa, hep-th/0312022
21. M. Kontsevich, alg-geom 9411018
22. M. Kontsevich, math/9709180
23. M. Kontsevich, math.QA/9904055
24. C. Korff, math-ph/0302002v2
25. M. Kontsevich, Y. Soibelman, math.QA/0001151v2
26. A. Klymnik, K. Schmüdgen, *Quantum Groups and Their Representations* (Springer, Berlin Heidelberg, 1997)
27. G. Lusztig, Introduction to quantum groups (Birkhäuser, Boston, 1993)
28. J. Madore, Class. Quantum Grav. **9**, 69 (1992)
29. A. Okounkov, N. Reshetikhin, C. Vafa, hep-th/0309208
30. S.-J. Rey, Phys. Rev. D **43**, 526 (1991)
31. K.-G. Schlesinger, J. Phys. A: Math. Gen. **35**, 10 189 (2002) [math.QA/0104275]
32. E. Witten, Mirror manifolds and topological field theory. In Essays on Mirror Manifolds, ed. by S.T. Yau (International Press, Hong Kong, 1991)
33. E. Witten, Surv. Differ. Geom. **1**, 243 (1991)